

SYMPLECTICALLY ASPHERICAL MANIFOLDS

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ABSTRACT. This is a survey article on symplectically aspherical manifolds.

1. INTRODUCTION

A symplectic form ω on a smooth manifold M is called *symplectically aspherical*, if for any smooth map $f : S^2 \rightarrow M$ one has the equality

$$(1.1) \quad \int_{S^2} f^* \omega = 0.$$

The latter condition is often written in the form $\omega|_{\pi_2(M)} = 0$. It is also useful to introduce a vector space

$$\Pi(M) = \text{Im}\{h : \pi_2(M) \rightarrow H_2(M)\} \otimes \mathbb{R},$$

where h is the Hurewicz map. In this notation, the symplectic asphericity condition can be written as

$$[\omega]|_{\Pi(M)} = 0, \quad \text{or } \langle [\omega], x \rangle = 0 \text{ for all } x \in \Pi(M)$$

where $[\omega]$ is the (de Rham) cohomology class of the form ω and $\langle -, - \rangle$ is the Kronecker pairing.

A symplectically aspherical manifold is, by definition, a manifold that is equipped with a symplectically aspherical form. Such manifolds were originally introduced and used by Floer [F] in order to attack the Arnold conjecture. The usefulness of symplectically aspherical manifolds comes from the theory of J -holomorphic curves and the Floer homology, which are easier for symplectically aspherical manifolds because of the absence of bubbling effects (see [F, H, HZ, MS1, MS2]). Since condition (1.1) is often imposed in many classical formulations of theorems in symplectic topology [LO, O], it is worthwhile to describe this class of manifolds. In the last decade a substantial understanding of symplectically aspherical manifolds was gained in [G1, IKRT, KRT, R2, RO, RT1, S]. However, there are still open interesting questions,

2000 *Mathematics Subject Classification.* Primary 53; Secondary 57.

Key words and phrases. Arnold conjecture; symplectically aspherical manifold; symplectically hyperbolic manifold.

and the whole subject becomes a rich mathematical theory involving many important topological and symplectic techniques. Motivated by this, we present a survey on recent developments in the theory of symplectically aspherical manifolds. We emphasize, however, that we are mainly concerned with the “soft” symplectic geometry, where the tools of algebraic topology are applicable. We use hard analytical tools (the pseudoholomorphic curves in Section 4, the applications of the action spectrum in Section 7) rather than develop them.

The main topics described in this article are the following.

- (1) Elementary topological properties of symplectically aspherical manifolds;
- (2) Lusternik–Schnirelmann category of closed symplectically aspherical manifolds;
- (3) Applications of symplectic asphericity to the classical Arnold conjecture on fixed points of symplectic diffeomorphisms;
- (4) Constructions of symplectically aspherical manifolds;
- (5) Discussion on fundamental groups of closed symplectically aspherical manifolds;
- (6) Applications of symplectic asphericity to the theory of the group of symplectic diffeomorphisms and to the symplectic action spectrum;
- (7) Discussion on symplectically hyperbolic manifolds;
- (8) Applications of symplectic asphericity to Lie group actions on closed symplectic manifolds.

We survey the known results and discuss research perspectives and conjectures.

Throughout this paper, all manifolds are assumed to be closed and connected (all exceptions are explicitly stated).

2. PRELIMINARIES ON SYMPLECTICALLY ASPHERICAL MANIFOLDS

In this section we present some facts which we use in the sequel. Throughout the section the manifolds are not assumed to be closed.

Proposition 2.1. *Let ω be a symplectically aspherical form on a manifold N , and let $g : M \rightarrow N$ be a map such that $g^*\omega$ is a symplectic form on a manifold M . Then $(M, g^*\omega)$ is symplectically aspherical. In particular, a covering manifold over a symplectically aspherical manifold is symplectically aspherical.*

Proof. Since the form ω on N is symplectically aspherical, we conclude that

$$\langle g^*\omega, h(a) \rangle = \langle \omega, g_*h(a) \rangle = \langle \omega, h(g_*(a)) \rangle = 0$$

for all $a \in \pi_2(M)$. Thus, $g^*\omega$ is symplectically aspherical. \square

We need the following homotopic characterization of symplectically aspherical manifolds. Given a group π , recall that the Eilenberg–Mac Lane space $K(\pi, 1)$ is a connected CW-complex with fundamental group π and such that $\pi_i(K(\pi, 1)) = 0$ for $i > 1$. It is well known that the homotopy type of such space is completely determined by π . Moreover, for every connected CW-space X with $\pi_1(X) = \pi$ there exists a map $f : X \rightarrow K(\pi, 1)$ that induces an isomorphism on the fundamental groups, and this map is unique up to homotopy. To construct such a map f , attach to X cells of dimensions > 2 in order to kill all the higher homotopy groups. The resulting space is $K(\pi, 1)$ and f is the inclusion of X .

Proposition 2.2 ([LO, Lemma 4.2], [RT, Corollary 2.2]). *Let (M, ω) be a symplectic manifold. The following three conditions are equivalent:*

- (i) *the form ω is symplectically aspherical;*
- (ii) *if a map $f : M \rightarrow K(\pi_1(M), 1)$ induces an isomorphism on the fundamental groups, then*

$$[\omega] \in \text{Im} \{f^* : H^2(K; \mathbb{R}) \rightarrow H^2(M; \mathbb{R})\};$$

- (iii) *there exist a group τ and a map $g : M \rightarrow K(\tau, 1)$ such that*

$$[\omega] \in \text{Im} \{f^* : H^2(K(\tau, 1); \mathbb{R}) \rightarrow H^2(M; \mathbb{R})\}.$$

In the context of symplectic asphericity, it seems natural to ask if there is a manifold M that possesses two symplectic forms ω_1 and ω_2 such that $\omega_1|_{\pi_2(M)} = 0$ and $\omega_2|_{\pi_2(M)} \neq 0$. The following Proposition answers this question.

Proposition 2.3. *Suppose that a closed manifold M admits a symplectic form. Then the following two conditions are equivalent:*

- (i) $\Pi(M) = 0$;
- (ii) *Every symplectic form on M is symplectically aspherical.*

Proof. Only the implication (ii) \implies (i) needs a proof. So, assume that $\Pi(M) \neq 0$ and consider a symplectic form ω on M . If $\omega|_{\Pi} \neq 0$ then we are done. So, assume that $\omega|_{\Pi} = 0$. Since $\Pi(M) \neq 0$, there exists a closed 2-form σ with $\langle [\sigma], x \rangle \neq 0$ for some $x \in \Pi(M)$, i.e. $\sigma|_{\pi_2(M)} \neq 0$. Now, the form $\gamma = \omega + \lambda\sigma$ is symplectic for λ small enough, and $\gamma|_{\pi_2(M)} \neq 0$. \square

3. LUSTERNIK–SCHNIRELMANN CATEGORY OF SYMPLECTICALLY ASPHERICAL MANIFOLDS

In this section we prove that, for any (closed) symplectically aspherical manifold M^{2n} , the number of critical points of any smooth function $f : M \rightarrow \mathbb{R}$ is at least $2n + 1$. We also describe some results that will be used in our discussion of the Arnold conjecture in Section 4.

3.1. Lusternik–Schnirelmann Theorem. Given a smooth function $F : M \rightarrow \mathbb{R}$ on a smooth manifold M , let $\text{crit } F$ denote the number of critical points of F . Put $\text{Crit } M = \min\{\text{crit } F\}$, where the minimum runs over all smooth functions $F : M \rightarrow \mathbb{R}$.

Definition 3.2 ([Fox]). Let $f : X \rightarrow Y$ be a map. The *Lusternik–Schnirelmann category* of f , denoted $\text{cat}(f)$, is defined to be the minimal integer k such that there exists an open covering $\{U_0, \dots, U_k\}$ of X with the property that each of the restrictions $f|_{U_i} : U_i \rightarrow Y$, $i = 0, 1, \dots, k$ is null-homotopic. If such a covering does not exist we say that $\text{cat}(f)$ is not defined.

The *Lusternik–Schnirelmann category* $\text{cat } X$ of a space X is defined as the category $\text{cat}(\text{Id}_X)$ of the identity map.

For the proof of the following Lusternik–Schnirelmann Theorem, see [CLOT].

Theorem 3.3. *For every closed manifold M we have the inequality $\text{Crit } M \geq \text{cat } M + 1$.*

This theorem admits the following generalization. Given a flow $\{\varphi_t : X \rightarrow X, t \in \mathbb{R}\}$, a rest point of the flow is defined as a point $x \in X$ such that $\varphi_t(x) = x$ for all $t \in \mathbb{R}$. The flow is *gradient-like* if there exists a function $F : X \rightarrow \mathbb{R}$ (called a *Lyapunov function*) such that $F(\varphi_t(x)) > F(\varphi_s(x))$ whenever $t < s$ and x is not a rest point of the flow.

Theorem 3.4. *Let X be a compact space and $f : X \rightarrow Y$ be a map such that $\text{cat}(f)$ is defined. Let $\{\varphi_t\}$ be a gradient-like flow on a compact space X . Then the number of rest points of the flow is at least $1 + \text{cat}(f)$.*

The proof can be found in [CLOT]. Note that Theorem 3.4 implies Theorem 3.3. Indeed, given a closed manifold M , let f be the identity map in Theorem 3.4. Now, given a smooth function $F : M \rightarrow \mathbb{R}$, we have the gradient-like flow – $\text{grad } F$ (this explains the name gradient-like), whose rest points are exactly the critical points of F .

3.5. Category weight. The following definition is a homotopy invariant version of a construction of Fadell–Husseini [FH]. It was suggested by Rudyak [R1] and Strom [St].

Definition 3.6. The *category weight* $\text{wgt}(u)$ of a non-zero cohomology class $u \in H^*(X; A)$ is defined as follows:

$$\text{wgt}(u) \geq k \iff \{\varphi^*(u) = 0 \text{ for every } \varphi: F \rightarrow X \text{ with } \text{cat}(\varphi) < k\}.$$

Proposition 3.7 ([R1, St, CLOT]). *Let A denote a coefficient ring. Category weight has the following properties.*

- (1) $1 \leq \text{wgt}(u) \leq \text{cat}(X)$, for all $u \in \tilde{H}^*(X; A)$, $u \neq 0$.
- (2) For every $f: Y \rightarrow X$ and $u \in H^*(X; A)$ with $f^*(u) \neq 0$ we have $\text{cat}(f) \geq \text{wgt}(u)$ and $\text{wgt}(f^*(u)) \geq \text{wgt}(u)$.
- (3) For $u \in H^*(X; A)$ and $v \in H^*(X; A)$ we have

$$\text{wgt}(u \cup v) \geq \text{wgt}(u) + \text{wgt}(v).$$

- (4) For every $u \in H^s(K(\pi, 1); A)$, $u \neq 0$, we have $\text{wgt}(u) = s$.
- (5) For every $u \in H^s(X; A)$, $u \neq 0$, we have $\text{wgt}(u) \leq s$.

3.8. Symplectic asphericity input. Here we show the effect of symplectic asphericity on the category weight of the symplectic class $[\omega]$.

Theorem 3.9 ([RO]). *If (M, ω) is a symplectically aspherical manifold then $\text{wgt}([\omega]) = 2$.*

Proof. It follows from Proposition 2.2 that $[\omega] = f^*a$ for some $a \in H^2(K(\pi_1(M), 1); \mathbb{R})$ and some $f: M \rightarrow K(\pi_1(M), 1)$. Now, $\text{wgt}(a) = 2$ by item (4) of Proposition 3.7, and thus $\text{wgt}([\omega]) = 2$ by items (2) and (5) of the same proposition. \square

Corollary 3.10. *If (M^{2n}, ω) is a symplectically aspherical manifold then $\text{cat } M = 2n$ and $\text{Crit } M = 2n + 1$.*

Proof. Since $[\omega]^n \neq 0$, we conclude that $\text{wgt}([\omega]^n) = 2n$ by Theorem 3.9 and items (3) and (5) of Proposition 3.7. Furthermore, $\text{cat } M \geq \text{wgt}([\omega]^n) = 2n$, and so $\text{Crit } M \geq 2n + 1$. Finally, according to Takens [T], we have $\text{Crit } N \leq \dim N + 1$ for every closed connected manifold N , and the result follows. \square

4. THE ARNOLD CONJECTURE

Let (M, ω) be a symplectic manifold. It is well-known that there exist a Riemannian metric g and an almost complex structure J on M such that $\omega(\xi, J\eta) = g(\xi, \eta)$.

4.1. Hamiltonian diffeomorphisms. Given a function $F : M \rightarrow \mathbb{R}$, define a symplectic-gradient vector field $\text{sgrad } F$ (frequently denoted also by X_F) by the condition

$$(4.1) \quad \omega(\text{sgrad } F, \xi) = -dF(\xi)$$

for all vector fields ξ . It is easy to see that $\text{sgrad } F = J \text{grad } F$ where $\text{grad } F$ is taken with respect to the metric g .

Now, consider a smooth function $H : S^1 \times M \rightarrow \mathbb{R}$ and put $H_t(x) = H(t, x)$. Consider the non-autonomic differential equation

$$(4.2) \quad \dot{x}(t) = \text{sgrad } H_t(x(t))$$

This equation yields a time-dependent flow $\Psi = \{\phi_t\} = \{\phi_t^H\}$ on M . Namely, if $x(t)$ is a solution of (4.2) with $x(0) = p \in M$, then $\phi_t(p) = x(t)$, $t \in \mathbb{R}$.

Definition 4.2. A diffeomorphism $\phi : M \rightarrow M$ is *Hamiltonian* if there exists a function (which is called a Hamiltonian) $H : S^1 \times M \rightarrow \mathbb{R}$ such that $\phi = \phi_1^H$. We also say that ϕ is a time-1 map of the Hamiltonian H . The set of all Hamiltonian diffeomorphism is denoted by $\text{Ham}(M, \omega)$.

The Arnold conjecture. *For every Hamiltonian diffeomorphism $\phi : M \rightarrow M$, the number of its fixed points is at least $\text{Crit } M$.*

Remark 4.3. There are several versions of the Arnold conjecture. The one above is the closest to the “soft” side of symplectic geometry. If we assume that a Hamiltonian diffeomorphism is non-degenerate, that is its graph intersects the diagonal transversely, then the conjecture claims that the number of fixed points is at least the number of critical points of any Morse function. This conjecture is neither proved nor disproved yet, even in the symplectically aspherical case. We mention here the papers [FO, LT] where the number of fixed points is estimated from below by the sum of Betti numbers.

4.4. Floer’s approach to the Arnold conjecture. Floer [F] suggested the following way to attack the Arnold conjecture for symplectically aspherical manifolds. If p is a fixed point of a Hamiltonian symplectomorphism $\phi = \phi_1^H$ then $\phi_t(p), t \in \mathbb{R}$ is a 1-periodic orbit, i.e. a loop $x : S^1 \rightarrow M$. Hence, we can count fixed points of ϕ by counting 1-periodic solutions of equation (4.2). So, we can try to pose a variational problem on the loop space of M whose solutions (extremals, critical loops) are the 1-periodic solutions of (4.2). Then we can apply the Lusternik–Schnirelmann theory to estimate of the number of extremals. Note that we will count only contractible 1-periodic orbits. Let us explain this in a bit more detail.

4.5. Variational reduction. Let $H : S^1 \times M \rightarrow \mathbb{R}$ be a Hamiltonian on a symplectically aspherical manifold (M, ω) . Given a contractible smooth loop $x : S^1 \rightarrow M$, we set

$$(4.3) \quad \mathcal{A}_H(x) = \int_{D^2} y^* \omega - \int_0^1 H(t, x(t)) dt$$

where $y : D^2 \rightarrow M$ is an extension of x . We call the functional \mathcal{A}_H on contractible loops the *symplectic action*. Note that the symplectic action is well-defined (it does not depend on the extension y) because of symplectic asphericity. So, we have the map

$$\mathcal{A}_H : C_c^\infty(S^1, M) := \{\text{contractible smooth maps } S^1 \rightarrow M\} \longrightarrow \mathbb{R}.$$

If we take the derivative of \mathcal{A}_H in the direction of the vector field ξ along $x(S^1)$ (regarded as a tangent vector to a loop x), we get

$$(4.4) \quad \begin{aligned} D\mathcal{A}_H(x)(\xi) &= \int_{S^1} \omega_{x(t)}(\dot{x}(t), \xi) - dH_t(x(t))\xi(t) dt \\ &= \int_{S^1} \omega_{x(t)}(\dot{x}(t) - \text{sgrad } H_t(x(t)), \xi) dt. \end{aligned}$$

So, if $x = x(t)$ is a critical orbit of \mathcal{A}_H , that is $D\mathcal{A}_H(x)(\xi) = 0$ for all ξ , then $\dot{x}(t) - \text{sgrad } H_t(x(t)) = 0$, (i.e. $x(t)$ is a 1-periodic solution of equation (4.2)).

In order to proceed, we must consider the “gradient flow” of \mathcal{A}_H . However, here we have many analytical difficulties that do not allow us to construct the gradient flows directly, cf. [HZ, Section 6.5]. Floer regards the gradient flow lines as maps

$$(4.5) \quad u : \mathbb{R} \times S^1 \rightarrow M, \quad (s, t) \mapsto u(s, t) = u(s, t + 1)$$

such that

$$(4.6) \quad \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} + \text{grad } H_t(u) = 0.$$

Floer [F] then obtained the following variational reduction theorem.

Theorem 4.6. *Let (M, ω) be a symplectically aspherical manifold. Assume also that $c_1(M)$ vanishes on $\pi_2(M)$. Let $\phi : M \rightarrow M$ be a Hamiltonian symplectomorphism. Then there exist a map $f : X \rightarrow M$ and a gradient-like flow Φ on X with the following properties:*

- (1) *X is a compact metric space.*
- (2) *The number of fixed points of ϕ is bounded from below by the number of rest points of Φ .*
- (3) *The map $f^* : H^*(M; A) \rightarrow H^*(X; A)$ is a monomorphism for any coefficient group A .*

Now we describe the data X, f and Φ of Theorem 4.6. Given a map $u : \mathbb{R} \times S^1 \rightarrow M$ and $\tau \in \mathbb{R}$, we define $u(\tau) : S^1 \rightarrow \mathbb{R}$, by $u(\tau)(s) = u(\tau, s)$.

Let X be a space of smooth maps $u : \mathbb{R} \times S^1 \rightarrow M$ such that

- (1) u satisfies the equation (4.6).
- (2) $u(0)$ (and hence $u(\tau)$ for all τ) is a contractible loop.
- (3) The function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, $\varphi(s) = \mathcal{A}_H(u(s))$ is bounded.

It turns out that X is compact. It is worth mentioning that the proof of compactness uses the symplectic asphericity of (M, ω) .

We define a flow $\Phi = \{\varphi_\tau, \tau \in \mathbb{R}\}$ on X by setting $\varphi_\tau(u(s, t)) = u(s + \tau, t)$. It can be proved that the flow is gradient-like with associated Lyapunov function

$$F : X \rightarrow \mathbb{R}, \quad F(u) = \mathcal{A}_H(u(0)).$$

Furthermore, if $u \in X$ is a rest point of Φ then $\partial u / \partial s = 0$. So, if we put $x = u(0) : S^1 \rightarrow M$ then

$$J(x) \frac{dx}{dt} + \text{grad } H_t(x) = 0,$$

or

$$\frac{dx}{dt} = J \text{grad } H_t(x) = \text{sgrad } H_t(x).$$

Note that the latter equation is obtained by applying J to both sides of the preceding equation and using $J^2 = -I$. So, x is a 1-periodic solution of the equation (4.2), and therefore $x(0)$ is a fixed point of the Hamiltonian diffeomorphism ϕ . Thus, the number of fixed points of ϕ is at least the number of rest points of Φ . Finally, we define a map $f :$

$X \rightarrow M$ by setting $f(u) = u(0, 0)$. The proof of the monomorphicity of f^* is difficult.

Remarks 4.7. 1. The above description of the space X is taken from the book [HZ]. In his original paper [F] Floer obtained the space X as a certain space of contractible loops $S^1 \rightarrow M$. As we have seen, critical points of \mathcal{A}_H are 1-periodic solutions of the equation (4.2). Generally, if one has a flow on a space, say, Y , one can consider a new space \overline{Y} whose points are the flow lines on Y , and define the flow on \overline{Y} via the time-shift. This describes the passages from Floer's interpretation to that of Hofer and Zehnder.

2. It seems that the condition $c_1(M)|_{\Pi(M)} = 0$ in Theorem 4.6 is redundant, cf. [HZ, Remark on p. 250], but as far as we know, nobody has written this down yet in the literature.

4.8. Proof of the Arnold conjecture for symplectically aspherical manifolds. Now, basing our argument on Theorem 4.6 and results of Section 3, we prove the Arnold conjecture under assumptions of Theorem 4.6.

Theorem 4.9 ([R2, RO]). *The Arnold conjecture holds for symplectically aspherical manifolds with $c_1(M)|_{\Pi(M)} = 0$.*

Proof. We use the notation from Theorem 4.6. It suffices to prove that the number of rest points of Φ is at least $2n + 1 = \text{Crit } M$. Since $f^*([\omega]^n) \neq 0$, we conclude that $\text{cat } f \geq \text{wgt}[\omega]^n = 2n$ by Proposition 3.7 and Theorem 3.9. So, by Theorem 3.4, the number of rest points of Φ is at least $2n + 1 = \text{Crit } M$. \square

4.10. Lagrangian submanifolds and their intersections. A Lagrangian submanifold of a symplectic manifold (V^{2n}, ω) is a smooth submanifold L^n of V such that $\omega|_L = 0$. Given a Hamiltonian diffeomorphism $\psi : V \rightarrow V$, the question on the number $\#(\psi(L) \cap L)$ of intersection points of $\psi(L)$ and L can be considered as a generalization of the Arnold conjecture. Indeed, given a symplectic manifold (M, ω) , the diagonal M of the symplectic manifold $(M \times M, \omega \times (-\omega))$ is Lagrangian. Furthermore, given a Hamiltonian diffeomorphism $\phi : M \rightarrow M$, define $\psi : M \times M \rightarrow M \times M$ as $\psi(x, y) = (\phi(x), y)$. Then the number $\#(\psi(M) \cap M)$ is exactly the number of fixed points of ϕ .

There is a large literature on Lagrangian intersections, we mention [EG, F, H]. In this context the symplectic asphericity appeared in [F, H] in the form $\pi_2(V, L) = 0$.

It is well known that the total space T^*L of the cotangent bundle of a smooth manifold L possesses a canonical symplectic form ω . Furthermore, the zero section L is a Lagrangian submanifold of (T^*L, ω) . Moyaux and Vandembroucq [MV] estimated from below the number $\#(\psi(L) \cap L)$ where $\psi : T^*L \rightarrow T^*L$ is a Hamiltonian diffeomorphism with a compact support. Generally this number is bounded by a certain numerical invariant $\text{Qcat } L$, but if L is symplectically aspherical then $\#(\psi(L) \cap L) \geq \text{Crit } L$, [CLOT, 8.5.1].

5. CONSTRUCTIONS OF SYMPLECTICALLY ASPHERICAL MANIFOLDS

In this section we present various constructions of symplectically aspherical manifolds. Mostly, they are based on the known constructions of symplectic manifolds which yield the symplectically aspherical property under additional hypotheses.

5.1. Branched coverings. Here we follow [G2]. Take a symplectically aspherical manifold (X, ω_X) , choose a symplectic submanifold $B \subset X$ of codimension 2 and construct (M, ω) as a covering of X branched along B . By construction the class $[\omega]$ of the symplectic form is the pull-back of the class $[\omega_X]$, and so ω is symplectically aspherical by Proposition 2.1. Note that Proposition 2.1 applies since outside the branch locus, we have a true covering.

By choosing the branching locus in a clever way, Gompf constructed the first examples of symplectically aspherical manifolds with non-trivial second homotopy group. Moreover, he proved that the symplectic asphericity and the vanishing of the first Chern class on spheres are independent conditions [G2, Theorem 7]). Also, Gompf constructed both Kähler and non-Kähler examples.

5.2. Homogeneous spaces. Let G be a simply connected solvable group and $\Gamma \subset G$ a uniform lattice. Recall that a uniform lattice is a discrete subgroup such that the quotient G/Γ is compact. Since a simply connected solvable group is contractible (in fact diffeomorphic to \mathbb{R}^n) the quotient is a $K(\Gamma, 1)$ -space. If G admits a Γ -invariant symplectic form then the quotient G/Γ is a symplectic manifold. This happens in many cases.

Let $\text{Lie}(G)$ denote the Lie algebra of a Lie group G . A group G is called completely solvable if the eigenvalues of the operators $\text{ad}_X : \text{Lie}(G) \rightarrow \text{Lie}(G)$ are real for any $X \in \text{Lie}(G)$. In this case the Hattori theorem [Ha] states that

$$H^*(G/\Gamma; \mathbb{R}) \cong H^*(\Lambda^*(\text{Lie}(G)); \mathbb{R}),$$

where the right-hand side is the cohomology of the Lie algebra of G . In particular, every cohomology class in $H^*(G/\Gamma; \mathbb{R})$ can be represented by a left-invariant form. Hence in order to show that G/Γ is symplectic it is enough to find a cohomology class $a \in H^2(G/\Gamma; \mathbb{R})$ such that its top power is non-zero. A manifold admitting such class a is called cohomologically symplectic. We summarize the above discussion in the following.

Theorem 5.3 ([IKRT, Lemma 4.2]). *Let G be a simply connected completely solvable Lie group and $\Gamma \subset G$ a uniform lattice. A homogeneous space $M = G/\Gamma$ is symplectically aspherical if and only if it is cohomologically symplectic.* \square

5.4. Symplectic bundles. A locally trivial bundle $(M, \omega) \rightarrow E \rightarrow B$ is called symplectic if its structure group is the group of symplectic diffeomorphisms of (M, ω) . The following theorem gives conditions

implying that the total space of a symplectic bundle is symplectically aspherical.

Theorem 5.5 ([IKRT, Theorem 7.4]). *Let (M, ω) and (B, ω_B) be symplectically aspherical manifolds. The total space of a symplectic bundle $(M, \omega) \xrightarrow{i} E \xrightarrow{p} B$ is symplectically aspherical if the following conditions are satisfied:*

- (1) *there exists $\Omega \in H^2(E)$ such that $i^*\Omega = [\omega]$ and*
- (2) *the class $[\omega]$ vanishes on $(i_*)^{-1}(\Pi(E))$.*

The first condition ensures the existence of a symplectic form on the total space. This was proved by Thurston in 1976, see [MS1, Theorem 6.3]. The second condition ensures the symplectic asphericity of the symplectic form coming from the Thurston construction.

5.6. Symplectic surgery. In [G1], Gompf proved that certain surgery can be performed symplectically. More precisely, let $j_i : (N, \omega_N) \rightarrow (M_i, \omega_i)$, $i = 1, 2$, be disjoint symplectic embeddings of codimension two. Suppose that their normal bundles have opposite Euler classes. Cut out small tubular neighborhood of the images $j_i(N)$ and glue the remaining part along the cutting locus to obtain a new manifold $X = M_1 \cup_N M_2$. This manifold admits a symplectic form which is equal to ω away from the gluing locus. The next theorem gives a condition under which this construction produces an aspherical symplectic form.

Theorem 5.7 ([IKRT, Theorem 6.3]). *Let $(M_1, \omega_1), (M_2, \omega_2), (N, \omega_N)$ be symplectic manifolds and $j_i : N \rightarrow M_i$ symplectic embeddings with opposite normal bundles. If M_1 and M_2 are symplectically aspherical and $(j_i)_* : \pi_1(N) \rightarrow \pi_1(M_i)$ are monomorphisms, then the Gompf symplectic sum $M_1 \cup_N M_2$ is symplectically aspherical.*

5.8. Lefschetz fibrations. An excellent exposition on Lefschetz pencils and fibrations can be found in [GS]. Let X be a compact, connected, oriented, smooth 4-manifold, possibly with boundary. A *Lefschetz fibration structure* on X is a surjective map $f : X \rightarrow \Sigma$, where Σ is a compact, connected, oriented surface and $f^{-1}(\partial\Sigma) = \partial X$. Furthermore, the following is required:

- the set $\{q_1, \dots, q_n\}$ of critical points of f is finite;
- $f(q_i) \neq f(q_j)$ for $i \neq j$;
- if $b \in \Sigma$ is a regular value of f then $f^{-1}(b)$ is a closed connected orientable surface;
- there exist an orientation-preserving complex charts $\varphi_i : U_i \rightarrow \mathbb{C}^2$ with $q_i \in U_i \subset X$ and $\psi_i : V_i \rightarrow \mathbb{C}$ with $f(q_i) \in V_i \subset$

$f(U_i) \subset \Sigma$ such that $\psi_i \circ f \circ \varphi_i^{-1} : \varphi(U_i) \rightarrow \mathbb{C}$ has the form $(x, y) \rightarrow x^2 + y^2$.

It is a celebrated result of Donaldson [D1] that a closed symplectic 4-manifold admits a structure of an oriented Lefschetz pencil, and, therefore, becomes a Lefschetz fibration after blowing up in a finite number of points. On the other hand, according to Gompf and Thurston, an oriented Lefschetz fibration admits a symplectic structure.

In principle, this gives a classification of 4-dimensional symplectic manifolds in terms of the monodromy of a Lefschetz pencil. One of the main results of [KRT] provides a condition under which a Lefschetz fibration admits an aspherical symplectic form. In the sequel we denote by Σ_g a closed orientable surface of genus g .

Theorem 5.9 ([KRT, Proposition 3.3]). *Let F be a closed connected orientable surface. Let $F \rightarrow X \rightarrow \Sigma_g$ be a symplectic Lefschetz fibration such that the inclusion of the fiber induces a non-trivial map*

$$H^2(X; \mathbb{R}) \rightarrow H^2(F; \mathbb{R}) = \mathbb{R}.$$

Put $Y = F \times \Sigma_h$ with the product symplectic structure. If $g+h > 0$ then the Gompf symplectic fiber sum $X \#_F Y$ is symplectically aspherical.

6. FUNDAMENTAL GROUPS OF SYMPLECTICALLY ASPHERICAL MANIFOLDS

For brevity, we call a group Γ *symplectically aspherical* if it can be realized as the fundamental group of a closed symplectically aspherical manifold.

Question 6.1. *What groups are symplectically aspherical?*

This question has various motivations. The first one belongs to a class of questions revolving around whether a given geometric structure imposes restrictions on the algebraic topology of the underlying manifold. The second motivation comes from the problem of describing properties of the fundamental group which determine the geometry of the manifold. In this section, we shall present results in the first direction.

An example of the second approach is Corollary 7.12.

Questions similar to 6.1 are still unanswered in the case of complex projective or Kähler manifolds [ABCKT]. It easily follows from the Lefschetz property that the first Betti number of the fundamental group of a closed Kähler manifold is even. According to Gromov and Shoen certain condition on the fundamental group of a closed Kähler manifold

M implies that it admits a holomorphic mapping onto a Riemann surface. Another example is the Shafarevich conjecture which states that if M is a complex manifold with $\pi_1(M)$ large, then the universal cover \widetilde{M} is a Stein manifold. Here the fundamental group is called *large* if for any non-constant holomorphic map $f : X \rightarrow M$ the image $f_*(\pi_1(X)) \subset \pi_1(M)$ is infinite.

Gompf proved in [G1] that every finitely presented group can be realized as the fundamental group of a closed symplectic manifold. In contrast, we shall show that it is not the case for symplectically aspherical manifolds. For example, observe that symplectically aspherical groups have to be infinite. Indeed, since the symplectic asphericity is preserved by finite coverings, if (M, ω) is symplectically aspherical and has finite fundamental group then its universal covering $(\widetilde{M}, \widetilde{\omega})$ is a symplectically aspherical simply connected closed manifold. This is impossible because a non-zero class $[\widetilde{\omega}]$ does not vanish on $H_2(\widetilde{M}; \mathbb{Z}) = \pi_2(\widetilde{M})$.

Proposition 6.2. *If a group Γ is a fundamental group of a symplectically aspherical manifold then either*

- (1) $\Gamma \cong \pi_1(\Sigma)$, where Σ is a closed oriented surface, or
- (2) there exists $\Omega \in H^2(\Gamma; \mathbb{R})$ with $\Omega^2 \neq 0$.

Proof. Let (M, ω) be a symplectically aspherical. The case of dimension 2 is trivial so let us assume that $\dim M > 2$. We know that $[\omega] = c^*\Omega$, where $c : M \rightarrow K(\pi_1(M), 1)$ is the classifying map. Since $[\omega]^2 \neq 0$ we get that $\Omega^2 \neq 0$. \square

In particular, no group of real cohomological dimension 3 is symplectically aspherical.

Constructions of symplectically aspherical groups are based on the constructions presented in Section 5. Using the Lefschetz fibrations it is possible to give a complete classification of symplectically aspherical Abelian groups.

Theorem 6.3 ([KRT, Theorem 1.2]). *A finitely generated Abelian group Γ is symplectically aspherical if and only if either $\Gamma \cong \mathbb{Z}^2$ or $\text{rank}(\Gamma) \geq 4$.*

As we mentioned in the beginning of this section, the first Betti number of a Kähler group is even. The next result states that there is no such restrictions in the symplectically aspherical case.

Theorem 6.4. *Any non-negative integer number that is different from one can be realized as the first Betti number of a symplectically aspherical manifold.*

Proof. **The case $b_1 = 0$.** Let Γ be a uniform lattice in $SU(2, 1)$. It acts freely by isometries on the complex hyperbolic plane and hence preserves the Kähler structure on it. Thus the quotient is an aspherical closed Kähler (and hence symplectic) manifold. Finally, $b_1(\Gamma) = 0$ [G, Ma].

The case $b_1 = 2$. The two dimensional torus serves as an example.

The case $b_1 = 3$. The Kodaira–Thurston manifold KT , see [MS1] is an example. More precisely, KT is the product $S^1 \times (N_3(\mathbb{R})/N_3(\mathbb{Z}))$, where $N_3(\mathbb{K})$ denotes the upper triangular matrices with coefficients in \mathbb{K} . Hence the fundamental group is isomorphic to $\mathbb{Z} \oplus N_3(\mathbb{Z})$ and it is easy to see that its first Betti number is equal to 3.

The case $b_1 \geq 4$. This follows from Theorem 6.3, or you can consider the products of tori and Kodaira–Thurston manifolds. \square

Remark 6.5. Unfortunately, we are unable to construct a symplectically aspherical manifold with the first Betti number equal to one. Nevertheless, we believe that such manifolds exist.

7. FURTHER APPLICATIONS OF SYMPLECTIC ASPHERICITY

7.1. The action spectrum. Let (M, ω) be a symplectic manifold and let $H : S^1 \times M \rightarrow \mathbb{R}$ be a Hamiltonian. A contractible solution $x : S^1 \rightarrow M$ of the equation (4.2) is called a contractible orbit. The set

$$\Sigma_H := \{\mathcal{A}_H(x) \mid x \text{ is a contractible orbit}\}$$

is called the action spectrum of H and is compact [HZ, S]. The following theorem of Schwarz is fundamental [S].

Theorem 7.2. *Let (M, ω) be a closed symplectically aspherical manifold. Let $H : S^1 \times M \rightarrow \mathbb{R}$ be a Hamiltonian whose time-1 map is not the identity. There are two contractible orbits $x, y \in M$ such that*

$$-\int_0^1 \max_M H_t dt \leq \mathcal{A}_H(x) < \mathcal{A}_H(y) \leq -\int_0^1 \min_M H_t dt.$$

Let $\psi \in \text{Ham}(M, \omega)$ be the time-1 map of a Hamiltonian $H : S^1 \times M \rightarrow \mathbb{R}$ and let $x, y \in M$ be its contractible orbits. Let $(\widetilde{M}, \widetilde{\omega})$ be the universal cover of (M, ω) and $\widetilde{\psi}$ be a lift of ψ . Since x, y are contractible orbits their lifts $\widetilde{x}, \widetilde{y} \in \widetilde{M}$ are contractible orbits as well (with respect to $\widetilde{H} : S^1 \times \widetilde{M} \rightarrow \mathbb{R}$ corresponding to $\widetilde{\psi}$). Take a curve $\gamma : [0, 1] \rightarrow \widetilde{M}$ with $\gamma(0) = \widetilde{x}(0)$ and $\gamma(1) = \widetilde{y}(0)$. Since \widetilde{M} is simply connected there exists a disc $u : D^2 \rightarrow \widetilde{M}$ with boundary $\partial u = \psi \circ \gamma - \gamma$. Define

$$\Delta(\psi; x, y) := \int_{D^2} u^*(\omega).$$

It is easy to see (using symplectic asphericity and the contractibility of the orbits) that the definition does not depend on the choice of γ and u .

The following lemma is proved by Polterovich [Po].

Lemma 7.3. *With the above notation the following hold:*

- (1) $\Delta(\psi^n; x, y) = n\Delta(\psi; x, y)$;
- (2) $\Delta(\psi; x, y) = \mathcal{A}_H(y) - \mathcal{A}_H(x)$.

As a simple corollary we get the following result.

Theorem 7.4. *Let (M, ω) be a symplectically aspherical manifold. Then the group $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms of (M, ω) is torsion-free.*

7.5. Hamiltonian representations of discrete groups. Let (M, ω) be a symplectically aspherical manifold. Let $\tilde{\omega} := p^*(\omega)$, where $p : \widetilde{M} \rightarrow M$ is the universal cover. Clearly, $\tilde{\omega}$ is an exact two-form. Let g be a Riemannian metric on M and \tilde{g} the pull-back metric on the universal cover. Choose a point $x_0 \in M$ and let $B(s)$ the ball of the radius s centered at x_0 . Let $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be defined by

$$(7.1) \quad u(s) := \inf_{d\alpha=\tilde{\omega}} \sup_{x \in B(s)} |\alpha(x)|_{\tilde{g}}.$$

The function $s \mapsto s \cdot u(s)$ is then strictly increasing. We define the *symplectic filling function* $v : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ to be its inverse.

If a_n and b_n are positive sequences then we write $a_n \succeq b_n$ if there exists a constant $c > 0$ such that $a_n \geq cb_n$ for all $n \in \mathbb{N}$, and we write $a_n \sim b_n$ if $a_n \succeq b_n$ and $b_n \succeq a_n$.

Let G be a finitely generated group equipped with the word metric with respect to some finite generating set A . Let $\|g\|_A$ denotes the distance of $g \in G$ from the identity. The following result is proved in [Po, Theorem 1.6.A].

Theorem 7.6. *Let (M, ω) be a symplectically aspherical manifold. Let A be a finite subset of $\text{Ham}(M, \omega)$ and G be the subgroup of $\text{Ham}(M, \omega)$ generated by A . Then $\|g^n\|_A \succeq v(n)$ for all $g \in G$.*

Definition 7.7. If the function $u : \mathbb{R} \rightarrow \mathbb{R}$ described in (7.1) is bounded, or equivalently $v \sim \text{Id}$, then the symplectic form ω is called *hyperbolic* and (M, ω) is called *symplectically hyperbolic*.

It is easy to see that boundedness of u does not depend on the choice of the metric g and the point x_0 .

Kähler manifolds of negative sectional curvature (e.g. closed surfaces of genus at least 2) are symplectically hyperbolic. The torus T^2 with the standard symplectic structure is not symplectically hyperbolic.

The interest in symplectically hyperbolic manifolds is, in particular, motivated by the following fact.

Corollary 7.8. *Let (M, ω) be a symplectically hyperbolic manifold. Let A be a finite subset of $\text{Ham}(M, \omega)$ and G be the subgroup of $\text{Ham}(M, \omega)$ generated by A . Then every cyclic subgroup $\langle g \rangle \subset G$ is undistorted with respect to the word metric given by A . In particular, $\text{Ham}(M, \omega)$ is torsion free.*

Remark 7.9. A cyclic subgroup $\langle g \rangle \subset G$ is undistorted if there exists a constant $C > 0$ such that $|g^n| \geq C \cdot n$, for all $n \in \mathbb{Z}$. Here $|x| := \min\{n \in \mathbb{N} \mid x = g_{i_1}^{p_1} \dots g_{i_k}^{p_k}, \sum_i p_i = n\}$ is the word norm of $x \in G$ with respect to a fixed finite set $\{g_1, \dots, g_m\}$ of generators of G . For example any cyclic subgroup of \mathbb{Z}^n is undistorted. Finite cyclic subgroups are not undistorted.

7.10. More symplectically hyperbolic manifolds. Here we provide a source of examples of symplectically hyperbolic manifolds. The subject is treated in more detail in a forthcoming paper [K]. Recall that a cohomology class is *bounded* if it is represented by a singular cochain whose values on singular simplices are uniformly bounded. Such cochains are called bounded as well.

Lemma 7.11. *Let (M, ω) be a symplectically aspherical manifold. If ω represents a bounded cohomology class then (M, ω) is symplectically hyperbolic.*

Proof. We just sketch the main ideas of proof referring to [K, Theorem 2.1] for details. Let $p : \widetilde{M} \rightarrow M$ be the universal cover. We show that $\tilde{\omega} := p^*(\omega) = d\alpha$, where α is a form bounded with respect to the metric \tilde{g} induced from any metric g on M .

Let $c \in C^2(M; \mathbb{R})$ be a bounded cochain representing the class $[\omega]$. Since \widetilde{M} is simply connected, we conclude that $p^*(c) = \delta(b)$ for some bounded real cochain $b \in C^1(\widetilde{M})$. Let K be a finite triangulation of M and K' the induced one of \widetilde{M} . Let b', c' be simplicial cochains corresponding to b and c respectively. The standard construction of a differential form from a simplicial cochain [STh, pages 148–149]) applied to b' gives a bounded form α whose differential is $p^*(\omega + d\beta)$ for some $\beta \in \Omega^1(M)$. Hence $p^*(\omega) = d(\alpha + p^*(\beta))$ while $\alpha + p^*(\beta)$ is clearly bounded. \square

Corollary 7.12. *Let (M, ω) be a symplectically aspherical manifold. If $\pi_1(M)$ is hyperbolic then ω is hyperbolic. In particular, if M admits a Riemannian metric of negative sectional curvature then ω is hyperbolic.*

Proof. By Proposition 2.2, $[\omega] = f^*(\Omega)$ for some $\Omega \in H^2(\pi_1(M); \mathbb{R})$ and $f : M \rightarrow K(\pi_1(M), 1)$. On the other hand, if $\pi_1(M)$ is hyperbolic then every cohomology class of $\pi_1(M)$ of degree greater than one is bounded. Hence $[\omega]$ is bounded as well and we apply Lemma 7.11. \square

Note that one can find the definition of a hyperbolic group in [Gr].

Corollary 7.13. *Let F be a closed oriented surface of genus at least 2, and let $F \xrightarrow{i} M \xrightarrow{p} B$ be an oriented bundle over a surface B of genus at least 1. Then M admits a hyperbolic symplectic form.*

Proof. Let ω_B be an area form on B . Given any class $\Omega \in H^2(M)$ with $i^*(\Omega) \neq 0$, the Thurston construction ([MS1, Theorem 6.3]) gives a symplectic form in the class $C \cdot p^*[\omega_B] + \Omega$, where $C > 0$ is a sufficiently large constant.

Let Ω be the Euler class of the bundle $V := \ker dp \rightarrow M$ tangent to the fibers of p . According to Morita [Mo] this class is bounded, and therefore $p^*[\omega_B] + \Omega$ is. \square

7.14. The Ostrover trick [Os]. Let $B \subset M$ be an open subset and let $h \in \text{Ham}(M, \omega)$ be a Hamiltonian diffeomorphism such that $h(B)$ is disjoint from the closure of B .

After slightly perturbing h we may assume that its fixed points are all non-degenerate. Let $F : M \times [0, 1] \rightarrow \mathbb{R}$ be a normalized Hamiltonian such that $F(x, t) = C$ for some $C < 0$ and $x \in M - B$, $t \in [0, 1]$. Take $\psi_t = h \circ f_t$ where f_t is the Hamiltonian flow generated by F .

Theorem 7.15. *Let (M, ω) be a symplectically aspherical manifold and let $\psi_t \in \text{Ham}(M, \omega)$ be as defined above. Then $\lim_{t \rightarrow \infty} d(\psi_t, \text{Id}) = \infty$. In particular, the Hofer diameter of $\text{Ham}(M, \omega)$ is infinite.*

Proof. Let F_t be a Hamiltonian for ψ_t and H a one associated to h . According to Schwarz, the Hofer norm of a Hamiltonian diffeomorphism is bounded from below by the minimum of the action spectrum. That is we have that

$$d(\psi_t, \text{Id}) \geq \min \Sigma_{F_t},$$

for all $t \in \mathbb{R}$. Notice that ψ_t and h have the same fixed points. Moreover, Ostrover proved (Proposition 2.6. in [Os]) that

$$\mathcal{A}_{F_t}(x) = \mathcal{A}_H(x) - t \cdot C.$$

Combining this with the previous inequality we get that the Hofer norm of ψ_t tends to infinity as t does. \square

8. APPLICATION OF SYMPLECTIC ASPHERICITY TO CIRCLE ACTIONS ON SYMPLECTIC MANIFOLDS

If a closed symplectic manifold (M, ω) admits a Lie group action preserving ω , the manifold must satisfy various topological restrictions. The nature of these restrictions is to some extent understood. It comes from Morse theory. For example, assume that (M, ω) admits a circle action that is *Hamiltonian*, i.e. $i_X\omega = d\mu$ for some smooth function $\mu : M \rightarrow \mathbb{R}$ and the fundamental vector field X determined by the circle action. This function μ is the *Bott-Morse* function, and this forces certain restrictions on the topology of M . Finding these conditions is now a huge research area. Restrictions on the equivariant cohomology of M are given in [Ki, TW], on characteristic classes, signature and Novikov numbers can be found in [Fa, Fe], restrictions on the topology of orbits in [Oz, Ko], on Massey products in [ST]. One can work in pure homotopic setting of *cohomologically symplectic manifolds* with circle actions, and still get non-trivial restrictions on the equivariant cohomology of M and the set of fixed points [A1, A2]. Many results in this theory are obtained as variations of the following fundamental fact. If G is a torus acting in a Hamiltonian way on a closed symplectic manifold M , then the fiber bundle

$$M \xrightarrow{i} EG \times_G M \longrightarrow BG$$

is *totally non-cohomologous to zero*, that is,

$$i^* : H_G^*(M; \mathbb{R}) := H^*(EG \times_G M; \mathbb{R}) \rightarrow H^*(M; \mathbb{R})$$

is onto [Ki]. In the sequel we will call this the TNCZ property. In general, TNCZ does not hold for cohomologically symplectic manifolds with circle actions [A1]. However, some properties of symplectic manifolds with circle actions do have cohomologically symplectic analogues. Such results follow as combinations of the localization theorem for equivariant cohomology with various additional algebraic assumptions. For example, if one imposes TNCZ condition on a circle action on a cohomologically symplectic manifold, one obtains that the set of fixed points of this action has at least two components, as in the case of true symplectic actions [A1, A2] (there are some additional technical assumptions, but we don't discuss them).

However, it seems that the natural boundaries of the theory are still not explored. For example, we don't know any examples of closed symplectic manifolds endowed with circle actions but whose topological properties differ from the ones established for symplectic manifolds with *symplectic* circle actions.

If one imposes the condition of symplectic asphericity, more restrictions can be found. In particular, Ono [O] found restrictions on the fundamental group of M in the presence of the symplectic asphericity condition. On the other hand, results of Lupton and Oprea [LO] suggest that these restrictions may have a purely homotopic nature.

8.1. Symplectic asphericity as an obstruction. Below we shall show how symplectic asphericity obstructs the existence of symplectic circle actions.

The following result is a weak version of [LO, Theorem 4.16] (see also [Op]).

Theorem 8.2. *Any S^1 -action on a symplectically aspherical manifold has no fixed points and hence is not Hamiltonian.*

Theorem 8.3. *Let (M, ω) be a closed symplectically aspherical manifold. If each Abelian subgroup of $\pi_1(M)$ is cyclic, then M does not admit non-trivial symplectic circle actions.*

Proof. First, it follows from the symplectic asphericity assumption and the previous theorem, that a circle action on M cannot be Hamiltonian. However, the action also cannot be non-Hamiltonian, which follows from the following argument. By way of contradiction assume that there exists a non-Hamiltonian action $a : S^1 \rightarrow \text{Symp}((M, \omega))$ and let X denote the vector field of this action. Since a is non-Hamiltonian, we conclude that $[i_X \omega] \neq 0$ in $H^1(M; \mathbb{R})$. Hence there exists a loop $A : S^1 \rightarrow M$ such that $\langle [i_X \omega], [A] \rangle \neq 0$ where $[A]$ is the homology class of the loop A . Now consider the map

$$a(A) : T^2 \rightarrow M, \quad a(A)(s, t) = a(s)(A(t)),$$

and it is easy to see that

$$\langle a(A)^*[\omega], [T^2] \rangle = \int_{T^2} a(A)^* \omega \neq 0.$$

Since the image of the homomorphism $a(A)_* : \pi_1(T^2) \rightarrow \pi_1(M)$ is cyclic, we conclude that there exists a simple loop C on T^2 whose homotopy class is nontrivial and belongs to the kernel of $a(A)_*$. Hence we get a map $f : S^2 \rightarrow M$ with $\langle [\omega], f_*[S^2] \rangle \neq 0$. But this contradicts the symplectic asphericity of (M, ω) . \square

Remark 8.4.

- (1) Theorem 8.3 was proved by Ono in [O].
- (2) It would be nice to find other topological restrictions on symplectically aspherical manifolds with circle actions, which follow

from the condition that the action is symplectic. It is conceivable, that Ono's condition is stronger: probably, it implies the non-existence of *smooth* circle actions on symplectically aspherical manifolds. A possible proof would go along the following lines. There is a notion of cohomologically Hamiltonian circle action [LO]. One could try to use it instead of the condition $[i_X\omega] = 0$. However, we did not work out the details.

- (3) In fact, [LO, Theorem 4.16] is a purely cohomological (and hence more general) version of Theorem 8.2.

8.5. On a problem of Taubes and related questions. The following problem was posed by Taubes [Ba], [FV].

Question 8.6. *Assume that a 4-manifold of the form $M^4 = N^3 \times S^1$ is symplectic. Is it true that N fibers over S^1 ?*

Note that if N fibers over S^1 then M admits a symplectic structure [Ba]. The question was answered in the affirmative in several important cases (see [FV]), but in general is still open. One can reformulate it in the following form.

Question 8.7. *Assume that (M^4, ω) is a closed symplectic manifold endowed with a free circle action. Does it admit a circle action preserving the given symplectic form ω ?*

It seems interesting to ask the similar question for manifolds of arbitrary dimension (which might be easier, since more classical differential topology methods are available). Note that the symplectic asphericity condition might help in looking for counterexamples in higher dimensions. Ono's theorem yields a justification: we know that any circle action on a symplectically aspherical (M, ω) cannot have fixed points, and we have a restriction on the fundamental group.

Acknowledgment. The research of the second author was supported by NSF grant 0406311. The third author was supported by the Ministry of Science and Higher Education of Poland, research project no. 1P03A 03330.

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